

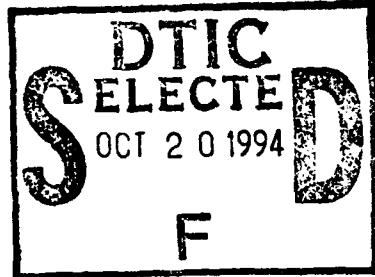
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## **THE VIRTUAL MASS AND LIFT FORCE ON AN OBLATE ELLIPSOID IN ROTATING AND STRAINING INVISCID FLOW**

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**Abstract:** The introduction of a rotating frame allows the use of a potential function to solve the problem of a rigid oblate ellipsoid undergoing rotation and translation in a rotating and straining inviscid flow. Virtual mass and lift force coefficients are calculated, including a coefficient associated with the ellipsoid rotation contribution to the lift force. Significantly, the computed coefficients can depart from sphere values.

**Key words:** *Virtual Mass, lift force, ellipsoid, inviscid flow.*

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### **1. Introduction.**

The virtual mass and the lift force on a sphere has been derived by several authors using inviscid flow theory. The original calculation of the so-called virtual mass force for a sphere is attributed to Lord Kelvin [Lamb, 1932]. This force arises when a rigid body accelerates through a quiescent fluid.

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The lift force *i.e.*, the force on a body moving relative to a straining fluid, has been calculated by Auton (1987) and Drew and Lahey (1987) for the particular case of a rigid sphere. In order to make the problem linear, Auton assumed that the velocity change on the sphere's surface due to the rotation of the fluid was much smaller than the relative translation velocity of the sphere. Drew and Lahey (1987) introduced a rotating frame of reference to eliminate the vorticity and to be able to use a scalar potential function to calculate the velocity field. Since the vorticity is a time-dependent magnitude, their solution is valid for the interval for which the vorticity in the rotating frame is small compared with the fluid vorticity at infinity in the inertial non-rotating system. Drew and Lahey (1990) obtained the same result than Auton's and demonstrated that their approach is more general.

When a gas bubble moves in a straining liquid it may undergo deformation. Thus, even if the bubble was initially a sphere, it will change its form to one that makes all the calculations mentioned above non-applicable, since the bubble is no longer a sphere nor a rigid body. Moreover, there is evidence that the failure to model bubbly flow after a sudden expansion is due to the fact that lift force modeling considers only rigid spherical bubbles [Bel Fdhila, 1991]. Despite its practical importance, very little attention has been given to the force over non-spherical bodies. The authors are aware of only the work of Lamb (1932), who calculated the virtual mass coefficient for an ellipsoid moving in a quiescent fluid.

It is the purpose of this report to calculate the virtual mass and lift force on a rigid oblate ellipsoid (*i.e.*, a ellipsoid obtained by the rotation of an ellipse around its minor

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semiaxis). Drew and Lahey's approach is used in section 2. In section 3 the velocity potential is calculated. Finally, in section 4 the force on an ellipsoid is obtained and from it, the appropriate coefficients are derived.

## 2. Statement of the problem and theoretical approach.

The problem to be solved is shown schematically in Figure-1. An oblate ellipsoid, whose center is, using indicial notation, at position  $x^*_{di}$  moves through an inviscid fluid with an unsteady translation velocity of  $v^*_{di}$  and rotates at a constant angular velocity of  $\omega^*_{di}$  (the superscript \* indicates the inertial system, the subscript  $d$  stands for the discrete phase and the dummy index  $i$  indicates the Cartesian components). The fluid far from the ellipsoid is undergoing a motion which consists of unsteady translation, a rotation at constant angular velocity and a constant strain. The velocity of the continuous phase far from the ellipsoid is thus

$$v^*_{ci} = v^*_{0i} + e^*_{ij} x^*_{ji} + \epsilon_{ijk} \omega^*_{0j} x^*_{ki} \quad (1)$$

where  $x^*_{ji}$  is the spatial coordinate,  $v^*_{0i}$  is the time-dependent undisturbed fluid velocity (the subscript 0 stands for undisturbed),  $e^*_{ij} = (v^*_{ci,j} + v^*_{cj,i})/2$  is the constant symmetric tensor representing the straining motion and  $\omega^*_{0i}$  is the constant angular rotation vector.

In order to get the velocity field from just a scalar velocity potential it is necessary to eliminate the vorticity. For that purpose a noninertial (unstarred) coordinate system which rotates with the fluid far from the ellipsoid is introduced. The systems are related by

$$x_i = Q_{ij} x^*_j \quad (2)$$

where  $Q_{ij}$  is an orthonormal rotation tensor which satisfies

$$Q_{ij} Q_{kj} = Q_{ji}, Q_{jk} = \delta_{ik} \quad (3)$$

and

$$\dot{Q}_{ij} = -Q_{ik} \epsilon_{kmj} \omega^*_{0m} \quad (4)$$

It can be concluded from Eq. (2) that

$$v_i = Q_{ij} v^*_j + \dot{Q}_{ij} x^*_j \quad (5)$$

For an incompressible inviscid fluid in a gravitational field a vorticity balance gives  
[Drew and Lahey, 1990]

$$\zeta_{j,i} + \zeta_{j,k} v_k - (2\omega_k + \zeta_k) v_{j,k} = 0 \quad (6)$$

where  $\zeta_j = \epsilon_{jik} v_{k,i}$  is the vorticity in the rotating unstarred system. From Eq. (6) it can be seen that the vorticity is not a constant magnitude, but depends on time. Thus the new coordinate system can have zero vorticity at a given instant, chosen arbitrarily to be  $t=0$ , but later it will grow. As a consequence the present analysis is valid only in the time scale that maintains the vorticity at a small enough value for the potential theory approximation to be satisfactory [Drew and Lahey, 1990].

In order to simplify the force calculation it is convenient to derive the potential in a non-rotating system. Thus, the following change of variables is introduced:

$$\hat{x}_i = Q_{ji} x_j \quad (7)$$

$$\hat{v}_i = Q_{ji} v_j \quad (8)$$

Notice that in the new (hat) system, the velocities are not obtained as the derivatives of the position. The hat and star system are related by

$$\hat{x}_i = x^*_i \quad (9)$$

$$\hat{v}_i = v^*_i + Q_{ji} \dot{Q}_{jk} x^*_k \quad (10)$$

Replacing Eq. (1) in Eq. (10) and using Eq. (4), we may conclude that:

$$\hat{v}_{oi} = v^*_{0i} + e^*_{ij} x^*_j \quad (11)$$

Thus the motion of the fluid far from the ellipsoid consists of an unsteady translation, at velocity  $\hat{v}_{oi} = v^*_{0i}$ , and a constant strain  $\hat{e}_{ij} = e^*_{ij}$ . The velocity on the ellipsoid's surface can be obtained in a similar manner:

$$\hat{v}_{di} = v^*_{di} + \epsilon_{ijk} (\omega^*_{0j} + \omega^*_{dj}) x^*_k \quad (12)$$

where, in the last equation,  $x^*_k$  belongs to the ellipsoid's surface and  $\omega^*_{dj}$  is the ellipsoid rotation in the inertial (starred) system. From the previous equations it is clear that an observer attached to the hat system will see the ellipsoid undergoing an unsteady translation at a velocity  $v^*_{di}$  and a rotation with an angular velocity  $\omega^*_{0j} + \omega^*_{dj}$ . This rotation means that the ellipsoid's surface is moving. This movement will make the boundary conditions on the ellipsoid's surface dependent on the surface position, except for the particular case in which  $\omega^*_{0j} + \omega^*_{dj}$  is parallel to the direction of the minor semiaxis of the ellipsoid. It will be seen later that there is an analytical solution for the velocity potential that satisfies the rotation boundary condition.

### 3. The velocity potential in the hat system.

The velocity potential,  $\phi$ , is defined by,

$$\hat{v}_i = \phi_i \quad (13)$$

and consequently, for an irrotational flow satisfies the Laplace equation,

$$\phi_{,ii} = 0 \quad (14)$$

Oblate spheroidal coordinates are defined by [Morse and Feshbach, 1953]:

$$\begin{aligned}\hat{x}_3 - \hat{x}_{d3} &= a \xi \eta \\ \hat{x}_1 - \hat{x}_{d1} &= a \sqrt{\xi^2 + 1} \sqrt{1 - \eta^2} \cos\theta \\ \hat{x}_2 - \hat{x}_{d2} &= a \sqrt{\xi^2 + 1} \sqrt{1 - \eta^2} \sin\theta\end{aligned} \quad (15)$$

where  $a > 0$  is a constant,  $\xi$  goes from 0 to infinity,  $\eta$  goes from -1 to 1 and  $\theta$  goes from  $-\pi$  to  $\pi$ . All the magnitudes and operators required to calculate the force on the ellipsoid in the oblate system of coordinates can be derived from these equations. The more important magnitudes are presented in the Appendix. The surfaces  $\xi = \xi_0 > 0$  are oblate ellipsoids with major and minor semiaxes given by,

$$b = a \sqrt{1 + \xi_0^2} \quad \text{and} \quad c = a \xi_0, \quad (16)$$

respectively. From the previous equations it can be seen that a sphere of radius  $R$  is the limiting case obtained when  $\xi_0$  goes to infinity while the product  $R = a\xi_0$  is kept constant. The surfaces  $\eta = \eta_0$  are hyperboloids of one sheet, asymptotic to the cone of angle  $\arccos(\eta)$  with respect to the  $\hat{x}_3$  axis, which is the axis of the cone.

The general solution for the Laplace equation in this coordinates system is [Morse and Feshbach, 1953]:

$$\phi(\xi, \eta, \theta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)] P_n^m(\eta) [C_{mn} P_n^m(i\xi) + D_{mn} Q_n^m(i\xi)] \quad (17)$$

where  $P_n^m$  and  $Q_n^m$  are the Legendre functions of first and second kind respectively,  $A_{mn}$ ,  $B_{mn}$ ,  $C_{mn}$  and  $D_{mn}$  are coefficients that depend only on indices  $m$  and  $n$  and,  $i = \sqrt{-1}$ . It must be noted that the coefficients  $C_{mn}$  and  $D_{mn}$  may be complex numbers as necessary to make the velocity potential a real function. To avoid this unnecessary complication, the Legendre functions used in this work were redefined in such a way that only real constant coefficients were needed. These Legendre functions are listed in the Appendix.

In order to calculate the potential, four cases are considered: In the first, the ellipsoid  $\xi = \xi_0 > 0$  is moving with velocity  $v^*_{di}$  and the fluid at infinity is at rest; in the

second, the ellipsoid is at rest and the fluid far away from the ellipsoid is moving with a velocity  $v^*_{\alpha i}$ ; in the third case, the velocity at infinity consists of the strain tensor and the ellipsoid is at rest; and in the last case, the ellipsoid is undergoing a rotation and the fluid at infinity is at rest. The general solution is formed by superposition of these four particular ones.

The boundary conditions for the first case are, on the ellipsoid's surface:

$$\phi_{,j} n_j = v^*_{,d_j} n_j \quad (18)$$

where  $n_j$  is the normal to the ellipsoid's surface, and at infinity:

$$\phi \xrightarrow{x \rightarrow \infty} 0 \quad (19)$$

In order to find the velocity potential, the velocity  $v^*_{di}$  is separated into its Cartesian components and the gradient is expressed in the oblate coordinate systems to obtain [Lamb, 1932]:

$$\begin{aligned} \phi = a & \left[ v^*_{,d_1} \cos \theta + v^*_{,d_2} \sin \theta \right] P_1^1(\eta) \frac{\xi_0}{\sqrt{1+\xi_0^2}} \left[ \frac{d}{d\xi} Q_1^1(i\xi) \right]_{\xi_0} + \\ & + a v^*_{,d_3} P_1^0(\eta) \left[ \frac{d}{d\xi} Q_1^0(i\xi) \right]_{\xi_0} \end{aligned} \quad (20)$$

The boundary condition for the second case, at infinity is:

$$\phi \xrightarrow{z \rightarrow \infty} v^*_{0j} \hat{x}_j \quad (21)$$

and at the ellipsoid's surface:

$$\phi_{,j} n_j = 0 \quad (22)$$

The solution can be found by replacing Eqs. (15) in Eq. (21) and comparing the result with Eq. (17). Since the functions  $Q_n^m(i\xi)$  tend asymptotically to zero when  $\xi$  tends to infinity, the functions  $P_n^m(i\xi)$  are the ones that satisfy the boundary condition at infinity and the functions  $Q_n^m(i\xi)$  are useful for the ellipsoid's surface boundary condition. Thus, the solution is:

$$\begin{aligned} \phi = & a [v^*_{01} \cos \theta + v^*_{02} \sin \theta] P_1^1(\eta) \left\{ P_1^1(i\xi) - \left[ \frac{d}{d\xi} P_1^1(i\xi) \right]_{\xi_0} Q_1^1(i\xi) \right\} + \\ & + a v^*_{03} P_1^0(\eta) \left\{ P_1^0(i\xi) - \left[ \frac{d}{d\xi} P_1^0(i\xi) \right]_{\xi_0} Q_1^0(i\xi) \right\} \end{aligned} \quad (23)$$

The boundary conditions for the third case are, at infinity:

$$\phi \xrightarrow{x \rightarrow \infty} \frac{1}{2} (\hat{x}_i - \hat{x}_{di}) e^*_{ij} (\hat{x}_j - \hat{x}_{dj}) \quad (24)$$

and at the ellipsoid's surface the boundary condition is the same as that for the second case, Eq. (22). The same procedure that was already used for the second case can be applied here to demonstrate that a solution that involves only a finite number of terms in Eq. (17) is possible only for the particular strain tensor given by,

$$e^* = \begin{pmatrix} 0 & 0 & e^*_{13} \\ 0 & 0 & e^*_{23} \\ e^*_{13} & e^*_{23} & 0 \end{pmatrix}, \quad (25)$$

and the solution is:

$$\begin{aligned} \phi = \frac{a^2}{9} [e^*_{13} \cos \theta + e^*_{23} \sin \theta] P_2^1(\eta) \left\{ P_2^1(i\xi) - \left[ \frac{d}{d\xi} P_2^1(i\xi) \right]_{\xi_0} \frac{Q_2^1(i\xi)}{\frac{d}{d\xi} Q_2^1(i\xi)} \right\} + \\ + a [e^*_{13} x^*_{d3} \cos \theta + e^*_{23} x^*_{d3} \sin \theta] P_1^1(\eta) \left\{ P_1^1(i\xi) - \left[ \frac{d}{d\xi} P_1^1(i\xi) \right]_{\xi_0} \frac{Q_1^1(i\xi)}{\frac{d}{d\xi} Q_1^1(i\xi)} \right\} + \\ + a [e^*_{23} x^*_{d2} + e^*_{13} x^*_{d1}] P_1^0(\eta) \left\{ P_1^0(i\xi) - \left[ \frac{d}{d\xi} P_1^0(i\xi) \right]_{\xi_0} \frac{Q_1^0(i\xi)}{\frac{d}{d\xi} Q_1^0(i\xi)} \right\} \end{aligned} \quad (26)$$

The boundary condition for the fourth case at the ellipsoid's surface comes from the second term of Eq. (12) and describes a rotation:

$$\phi_{,j} n_j = \epsilon_{ijk} (\omega^*_{0j} + \omega^*_{dj}) \hat{x}_k n_i \quad (27)$$

The condition at infinity is the same as for the first case (see Eq. (19)). Assuming that the ellipsoid's minor semiaxis is parallel to the  $\hat{x}_3$  axis,  $\omega^*_{0j} + \omega^*_{dj}$  can be expressed in terms of the oblate unit vectors and the right hand side of (27) expressed in terms of the oblate coordinate system. It would seem that the expression obtained in this way is valid only at the instant for which the ellipsoid minor semiaxis is parallel to the  $\hat{x}_3$  axis, because later the ellipsoid's rotation will change the expression of  $\omega^*_{0j} + \omega^*_{dj}$  in terms of the oblate unit vectors. However, since  $\omega^*_{0j} + \omega^*_{dj}$  is parallel to the axis of rotation of the ellipsoid, its projection on the oblate unit vectors at a given point over the surface does not change after a rotation. Thus, the expression obtained assuming a particular orientation of the ellipsoid with respect to  $\omega^*_{0j} + \omega^*_{dj}$  is completely general. The result is:

$$\varepsilon_{ijk} (\omega^*_{0j} + \omega^*_{dj}) \hat{x}_k n_i = \frac{a [ -(\omega^*_{01} + \omega^*_{d1}) \sin \theta + (\omega^*_{02} + \omega^*_{d2}) \cos \theta ] \eta \sqrt{1 - \eta^2}}{\sqrt{\xi_0^2 + \eta^2}} \quad (28)$$

As expected, the  $\omega^*_{03} + \omega^*_{d3}$  component does not appear in equation (28). Since the  $\hat{x}_3$  axis is a rotational symmetry axis, when the ellipsoid rotates around the  $\hat{x}_3$  axis it does not displace liquid. Thus, for this particular rotation the boundary condition is similar to that of the sphere which is independent of its rotation. Combining Eqs. (28) and (27), and expressing the left hand side of this last equation in the oblate coordinate system, the solution for this case is obtained as,

$$\phi = \frac{a^2}{3} [ -(\omega_{o_1}^* + \omega_{d_1}^*) \sin \theta + (\omega_{o_2}^* + \omega_{d_2}^*) \cos \theta ] P_2^1(\eta) \frac{1}{\sqrt{\xi_0^2 + 1}} \left[ \frac{d}{d\xi} Q_2^1(i\xi) \right]_{\xi_0} \quad (29)$$

Interestingly, Lamb first derived this solution in 1932.

The final solution for the problem is obtained by adding the four particular ones. However, since the solutions for the first three cases assume an orientation of the ellipsoid with respect to the inertial system, in order to apply the superposition principle to build a general solution,  $\nu_{o_i}^*$ ,  $\nu_{d_i}^*$  and  $e_{ij}^*$  have to be projected to a frame of reference attached to the ellipsoid. This is accomplished by replacing  $\nu_{o_i}^*$ ,  $\nu_{d_i}^*$  and  $e_{ij}^*$  in the already derived solutions for the four particular cases (*i.e.*, Eqs. (20),(23),(26) and (29)) by,

$$\hat{Q}_{ij} \nu_{o_j}^*, \quad (30)$$

$$\hat{Q}_{ij} \nu_{d_j}^* \quad (31)$$

and,

$$\hat{Q}_{ik} \hat{Q}_{jl} e_{ij}^* \quad (32)$$

respectively, where,

$$\dot{Q}_{ij} = -\hat{Q}_{ik} \epsilon_{kmj} (\omega^*_{om} + \omega^*_{dm})$$

with,

$$\hat{Q}_{ij}(t=0) = \delta_{ij}$$

In order to simplify the notation, from now on  $\hat{Q}_{ij}$  will be dropped from the equations. Thus, the forces to be calculated in the next section will be valid at the initial instant  $t=0$ . To obtain the forces at any moment later,  $v^*_{oi}$ ,  $v^*_{di}$  and  $e^*_{ij}$  will have to be replaced by their values given in Eqs. (30) to (32). With all these qualifications, the final velocity potential can be written as,

$$\begin{aligned}
\phi = & a[v^*_{d1} \cos \theta + v^*_{d2} \sin \theta] P_1^1(\eta) \frac{\xi_0}{\sqrt{1+\xi_0^2}} \left[ \frac{d}{d\xi} Q_1^1(i\xi) \right]_{\xi_0} + a v^*_{d3} P_1^0(\eta) \left[ \frac{d}{d\xi} Q_1^0(i\xi) \right]_{\xi_0} + \\
& + a[(v^*_{01} + x^*_{d3} e^*_{13}) \cos \theta + (v^*_{02} + x^*_{d3} e^*_{23}) \sin \theta] P_1^1(\eta) \left\{ P_1^1(i\xi) - \left[ \frac{d}{d\xi} P_1^1(i\xi) \right]_{\xi_0} Q_1^1(i\xi) \right\} + \\
& + a(v^*_{03} + x^*_{d1} e^*_{13} + x^*_{d2} e^*_{23}) P_1^0(\eta) \left\{ P_1^0(i\xi) - \left[ \frac{d}{d\xi} P_1^0(i\xi) \right]_{\xi_0} Q_1^0(i\xi) \right\} + \\
& + \frac{a^2}{9}[e^*_{13} \cos \theta + e^*_{23} \sin \theta] P_2^1(\eta) \left\{ P_2^1(i\xi) - \left[ \frac{d}{d\xi} P_2^1(i\xi) \right]_{\xi_0} Q_2^1(i\xi) \right\} + \\
& + \frac{a^2}{3}[-(\omega^*_{01} + \omega^*_{d1}) \sin \theta + (\omega^*_{02} + \omega^*_{d2}) \cos \theta] P_2^1(\eta) \frac{1}{\sqrt{\xi_0^2 + 1}} \left[ \frac{d}{d\xi} Q_2^1(i\xi) \right]_{\xi_0}
\end{aligned} \tag{33}$$

#### 4. Derivation of the force on an ellipsoid.

The momentum balance in the inertial system is [Morse and Feshbach, 1953]:

$$-\frac{p^*_j}{\rho} = v^*_{i,i} + \frac{1}{2}(v^*_{,j} v^*_{,j})_{,i} - \epsilon_{ijk} v^*_{,j} \zeta^*_{,k} \tag{34}$$

where  $\rho$  is the continuous phase density.

Integration of the previous equation along a current line from infinity to the ellipsoid's surface  $\xi = \xi_0$  yields:

$$-\frac{(p^*(\infty) - p^*(\xi_0, \eta, \theta))}{\rho} = \int_{\xi_0}^{\xi} v^*_{is} dl^*_i + \frac{1}{2} [(\nu^*_{ij} \nu^*_{ji})(\infty) - (\nu^*_{ij} \nu^*_{ji})(\xi_0, \eta, \theta)] \quad (35)$$

where  $dl^*_i$  is the arc length differentials along the current line. It can also be seen that the integral of the last term in Eq. (34) drops out because  $\epsilon_{ijk} \nu^*_{ij} dl^*_k \equiv 0$ .

In order to evaluate the right hand side of Eq. (35), it is convenient to express the velocity of the inertial system as a function of that in the hat system. In particular, from Eq. (10) and (4), and the facts that  $\omega^*_{ij}$  is constant and the time derivative is partial, it is concluded that,

$$\nu^*_{is} = \hat{v}_{is} \quad (36)$$

The velocity potential was derived with the ellipsoid at the origin. Thus, the velocity potential is a function,  $f$ , of  $\hat{x}_i - \hat{x}_{di}$  and  $t$ :

$$\phi = f(\hat{x}_i - \hat{x}_{di}(t), t) \quad (37)$$

where the potential depends explicitly on time through  $\hat{v}_{0i}$  and  $\hat{v}_{di}$ . It is concluded from Eqs. (13), (36), (37) and (9) that,

$$v^*_{i,t} = (\phi_s - \phi_{,j} v^*_{dj})_i \quad (38)$$

From this last relationship it is readily seen that,

$$\int_{\xi_0}^{\infty} v^*_{i,s} dl^*_i = (\phi_s - \phi_{,j} v^*_{dj})(\infty) - (\phi_s - \phi_{,j} v^*_{dj})(\xi_0, \eta, \theta) \quad (39)$$

After replacement of Eq. (39) in Eq. (35) and rearrangement, the following relationship is obtained:

$$0 = \left[ -\frac{p^*}{\rho} - (\phi_s - \phi_{,j} v^*_{dj}) - \frac{1}{2} v^*_{,j} v^*_{,j} \right] (\infty) = \left[ -\frac{p^*}{\rho} - (\phi_s - \phi_{,j} v^*_{dj}) - \frac{1}{2} v^*_{,j} v^*_{,j} \right] (\xi_0, \eta, \theta) \quad (40)$$

where the pressure at infinity was chosen in such a way that Eq. (40) holds.

Before integration of the last equation over the ellipsoid's surface, it is convenient to use Eqs. (10) and (4) to obtain:

$$v^*_{,j} v^*_{,j} = \hat{v}_j \hat{v}_j + 2 \epsilon_{jk} \omega^*_{0k} x^*_{,l} \hat{v}_j + \epsilon_{jk} \epsilon_{jm} \omega^*_{0k} x^*_{,l} \omega^*_{0m} x^*_{,n} \quad (41)$$

The net force on the ellipsoid is then obtained as the integral of Eq. (40) over the ellipsoid's surface:

$$-\iint p^* n_i dS = \rho \iint \phi_i n_i dS - \rho \iint \phi_j v^*_{dj} n_i dS + \frac{\rho}{2} \iint v^*_{ij} v^*_{ji} n_i dS \quad (42)$$

It should be noted that  $n_i$  is the unit normal at the ellipsoid's surface. The area differential  $dS$  can be found in the Appendix. The last integral in Eq. (42) can be expanded using Eq. (41):

$$\frac{1}{2} \iint v^*_{ij} v^*_{ji} n_i dS = \frac{1}{2} \iint \hat{v}_j \hat{v}_i n_i dS + \iint \epsilon_{jk} \omega^*_{0k} (x^*_{ik} - x^*_{dk}) \hat{v}_j n_i dS \quad (43)$$

The integral over the ellipsoid's surface of the last term in the right hand side of Eq. (41) is zero because

$$\iint (x^*_{ik} - x^*_{dk}) (x^*_{in} - x^*_{dn}) n_i dS = 0, \quad (44)$$

as can be demonstrated multiplying the integral by a constant vector and applying the divergence theorem to the result. This result is an extension of a similar one used for a sphere [Voinov, 1973].

The following result for the net force on the ellipsoid is obtained by combining Eqs. (43) and (42):

$$-\iint p^* n_i dS = \rho \iint \phi_{,i} n_i dS - \rho \iint \phi_{,j} v^*_{,ij} n_i dS + \frac{\rho}{2} \iint \hat{v}_j \hat{v}_j n_i dS + \rho \iint \epsilon_{ijk} \omega^*_{0k} (x^*_{,i} - x^*_{,k}) \hat{v}_j n_i dS \quad (45)$$

Using the velocity potential function,  $\phi$ , calculated in the previous section, all the integrals in the last equation can be evaluated. To make the exposition of results simpler, from now on it will be assumed that  $x_{di} \equiv 0$ . To obtain the force for the case  $x_{di} \neq 0$ , the term  $x^*_{,k} e^*_{,k}$  has to be added to  $v^*_{0k}$ . The results are:

$$\begin{aligned} \iint \phi_{,i} n_i dS &= \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{\beta}{\beta - 2} (v^*_{di,t} - v^*_{0i,t}) + v^*_{0i,t} \right] \quad (i = 1, 2) \\ &= \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{\beta - 1}{\beta} (v^*_{di,t} - v^*_{0i,t}) + v^*_{0i,t} \right] \quad (i = 3), \end{aligned} \quad (46)$$

$$\iint \phi_{,j} v^*_{,ij} n_i dS = \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] e^*_{ik} v^*_{,ik} \quad (47)$$

$$\begin{aligned}
\frac{1}{2} \iint \hat{v}_j \hat{v}_i n_i dS &= \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ -\frac{(3\beta - 2)}{2\beta} \epsilon_{ijk} (\omega^{*}_{0j} + \omega^{*}_{0k}) (\nu^{*}_{d3} - \nu^{*}_{03}) + \right. \\
&\quad \left. + \frac{\beta - 1}{\beta} e^{*}_{ik} (\nu^{*}_{dk} - \nu^{*}_{0k}) + e^{*}_{ik} \nu^{*}_{0k} \right] \quad (i = 1, 2) \\
&= \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ -\frac{(3\beta - 2)}{(\beta - 2)} \epsilon_{ijk} (\omega^{*}_{0j} + \omega^{*}_{0k}) (\nu^{*}_{dk} - \nu^{*}_{0k}) + \right. \\
&\quad \left. + \frac{\beta}{(\beta - 2)} e^{*}_{ik} (\nu^{*}_{dk} - \nu^{*}_{0k}) + e^{*}_{ik} \nu^{*}_{0k} \right] \quad (i = 3)
\end{aligned} \tag{48}$$

and,

$$\begin{aligned}
\iint \epsilon_{jkl} \omega^{*}_k (x^{*}_l - x^{*}_d) \hat{v}_j n_i dS &= \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ -\frac{\beta}{(\beta - 2)} \epsilon_{i3l} \omega^{*}_{03} (\nu^{*}_{dl} - \nu^{*}_{0l}) - \right. \\
&\quad \left. + \frac{1}{2} \epsilon_{ik3} \omega^{*}_{0k} (\nu^{*}_{d3} - \nu^{*}_{03}) - \epsilon_{ikl} \omega^{*}_{0k} \nu^{*}_{0l} \right] \quad (i = 1, 2) \\
&= \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{2(\beta - 1)}{(\beta - 2)} \epsilon_{ikl} \omega^{*}_{0k} (\nu^{*}_{dl} - \nu^{*}_{0l}) - \right. \\
&\quad \left. - \epsilon_{ikl} \omega^{*}_{0k} \nu^{*}_{0l} \right] \quad (i = 3)
\end{aligned} \tag{49}$$

where  $\left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right]$  is the ellipsoid volume, and,

$$\beta = \xi_0 \left( \frac{\xi^2}{\xi_0} + 1 \right) \arctan \left( \frac{1}{\xi_0} \right) - \xi_0^2 \quad (50)$$

It is worth noting that the integral in Eq. (48) is the only one where the ellipsoid rotation,  $\omega_{di}^*$ , provides a non-zero contribution to the force. From Eqs. (45) to (49), the net force on the ellipsoid can be written as,

$$\begin{aligned}
-\iint p^* n_i dS &= \rho \left[ \frac{4}{3} \pi \alpha^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{\beta}{2 - \beta} (\nu^*_{0i,t} - \nu^*_{di,t}) + \frac{1 - \beta}{\beta} e^*_{ik} (\nu^*_{0k} - \nu^*_{dk}) \right. \\
&\quad \left. - \frac{\beta}{2 - \beta} \varepsilon_{ijl} \omega^*_{03} (\nu^*_{0l} - \nu^*_{dl}) - \frac{1 - \beta}{\beta} \varepsilon_{ik3} \omega^*_{0k} (\nu^*_{03} - \nu^*_{d3}) \right] \\
&\quad - \left( \frac{1}{2} - \frac{1 - \beta}{\beta} \right) \varepsilon_{ij3} \omega^*_{dl} (\nu^*_{d3} - \nu^*_{03}) \Big] + \\
&+ \rho \left[ \frac{4}{3} \pi \alpha^3 \xi_0 (\xi_0^2 + 1) \right] [\nu^*_{0i,t} - e^*_{ik} \nu^*_{dk} + e^*_{ik} \nu^*_{0k} - \varepsilon_{ik} \omega^*_{0k} \nu^*_{0l}] \quad (i = 1, 2) \\
&= \rho \left[ \frac{4}{3} \pi \alpha^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{1 - \beta}{\beta} (\nu^*_{0i,t} - \nu^*_{di,t}) + \frac{\beta}{2 - \beta} e^*_{ik} (\nu^*_{0k} - \nu^*_{dk}) \right. \\
&\quad \left. - \frac{\beta}{2 - \beta} \varepsilon_{ijk} \omega^*_{0j} (\nu^*_{0k} - \nu^*_{dk}) - \left( 1 - 2 \frac{\beta}{2 - \beta} \right) \varepsilon_{ijk} \omega^*_{dl} (\nu^*_{dk} - \nu^*_{0k}) \right] \\
&+ \rho \left[ \frac{4}{3} \pi \alpha^3 \xi_0 (\xi_0^2 + 1) \right] [\nu^*_{0i,t} - e^*_{ik} \nu^*_{dk} + e^*_{ik} \nu^*_{0k} - \varepsilon_{ik} \omega^*_{0k} \nu^*_{0l}] \quad (i = 3)
\end{aligned} \tag{51}$$

**It is convenient to note that,**

$$\rho \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] [v^*_{0,i} - e^*_{ik} v^*_{ik} + e^*_{ik} v^*_{0k} - \varepsilon_{ik} \omega^*_{0k} v^*_{0i}] = \\ - \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] p^*_{0i} \quad (52)$$

where  $p^*_0$  is the pressure in the undisturbed fluid. The first two terms in Eq. (51) are the virtual mass force,

$$\begin{aligned} \left( \begin{array}{c} \text{Virtual mass} \\ \text{force} \end{array} \right)_i &= \rho \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{\beta}{2-\beta} (v^*_{0i,i} - v^*_{ik,i}) + \frac{1-\beta}{\beta} e^*_{ik} (v^*_{0k} - v^*_{ik}) \right] \\ &\quad (i = 1, 2) \\ &= \rho \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ \frac{1-\beta}{\beta} (v^*_{0i,i} - v^*_{ik,i}) + \frac{\beta}{2-\beta} e^*_{ik} (v^*_{0k} - v^*_{ik}) \right] \\ &\quad (i = 3) \end{aligned} \quad (53)$$

Lamb (1932) obtained this result for the particular case  $e^*_{ik} = 0$ . Since  $\beta$  tends to 2/3 when  $\xi_0$  goes to infinity, the well known coefficient value of 1/2 [Auton, 1987] [Drew and Lahey, 1987] is recovered for the sphere limit. The coefficients  $\beta/(2-\beta)$  and  $(1-\beta)/\beta$  can be seen in Figure- 2. To interpret this figure it is convenient to consider the following relationship, which can be derived from Eqs. (16),

$$\xi_0 = \left[ \left( \frac{b}{c} \right)^2 - 1 \right]^{-1/2} \quad (54)$$

As expected, for the special case of  $e^*_{ik} \equiv 0$ , the virtual mass coefficient in the direction  $i=3$  is always greater than for the directions  $i=1,2$ . This results because the area projected by the ellipsoid into a plane perpendicular to the direction of movement is proportional to  $b^2$  for  $i=3$  and to  $bc < b^2$  for  $i=1,2$ . Thus, the amount of fluid displaced in the former case has to be greater than in the latter. Since for both directions,  $i=1,2$ , the virtual mass coefficient is the same (when  $e^*_{ik} \equiv 0$ ), the problem of obtaining the virtual mass coefficient for an arbitrary direction is bidimensional. Thus, the coefficient of virtual volume,  $C_{vm}$ , when the relative acceleration of the ellipsoid,  $-(v^*_{oi,t} - v^*_{di,t})$ , forms an angle  $\varphi$  with the  $x^*_3$  axis is,

$$C_{vm}(\varphi) = \sqrt{\left(\frac{1-\beta}{\beta}\right)^2 \cos^2 \varphi + \left(\frac{\beta}{2-\beta}\right)^2 \sin^2 \varphi} \quad (55)$$

A similar definition is obtained for the case  $(v^*_{oi,t} - v^*_{di,t}) \equiv 0$  and  $e^*_{ik} \neq 0$ .

Even though the virtual mass coefficient  $(1-\beta)/\beta$  tends to infinity when  $\xi_0$  tends to zero, the total virtual mass force is not infinite because, if  $a$  is kept constant, the ellipsoid volume tends to zero. The total force, which corresponds to a flat disk of radius  $a$ , can be obtained from the following limit,

$$\left[ \frac{4}{3}\pi a^3 \xi_0 (\xi_0^2 + 1) \right] \frac{1-\beta}{\beta} \xrightarrow{\xi_0 \rightarrow 0} \frac{8}{3} a^3 \quad (56)$$

When the ellipsoid volume is kept constant, the total force and  $\alpha$  go to infinity when  $\xi_0$  tends to zero. Since  $\alpha$  is the disk radius, the total force needed to accelerate the disk has to diverge.

The last two terms in Eq. (51) are the lift force,

$$\begin{aligned}
 (\text{Lift force})_i &= \rho \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ -\frac{\beta}{2-\beta} \varepsilon_{i3l} \omega^*_{03} (\nu^*_{0l} - \nu^*_{d3}) \right. \\
 &\quad \left. - \frac{1-\beta}{\beta} \varepsilon_{ik3} \omega^*_{0k} (\nu^*_{03} - \nu^*_{d3}) - \left( \frac{1}{2} - \frac{1-\beta}{\beta} \right) \varepsilon_{ij3} \omega^*_{dj} (\nu^*_{d3} - \nu^*_{03}) \right] \quad (i = 1, 2) \\
 &= \rho \left[ \frac{4}{3} \pi a^3 \xi_0 (\xi_0^2 + 1) \right] \left[ -\frac{\beta}{2-\beta} \varepsilon_{ijk} \omega^*_{0j} (\nu^*_{0k} - \nu^*_{dk}) - \right. \\
 &\quad \left. - \left( 1 - 2 \frac{\beta}{2-\beta} \right) \varepsilon_{ijk} \omega^*_{dj} (\nu^*_{dk} - \nu^*_{0k}) \right] \quad (i = 3)
 \end{aligned} \tag{57}$$

Taking the limit when  $\beta$  tends to  $2/3$ , it can readily be shown that the sphere's lift force is recovered from the last equation. There are two different contributions in the last equation, one coming from the vorticity far away from the ellipsoid,  $\omega^*_\alpha$ , and the other

coming from the ellipsoid's rotation,  $\omega^*_{di}$ . Let us consider the former contribution first:

When the relative velocity  $-(v^*_{0i} - v^*_{di})$  is in the direction  $i=3$  (or equivalently  $\omega^*_{\theta i}$  is in the direction  $j=1,2$ ), the lift force coefficient is  $(1-\beta)/\beta$ . Otherwise the coefficient is  $\beta/(2-\beta)$ . It must be noted that, due to the same reason as in the virtual mass case, the bigger lift force coefficient,  $(1-\beta)/\beta$ , occurs when the projected area of the ellipsoid into a plane perpendicular to the relative velocity is the greater and the smaller lift force coefficient,  $\beta/(2-\beta)$ , occurs in the opposite situation. The problem of determining the lift force coefficient for an ellipsoid moving with a relative velocity,  $-(v^*_{0i} - v^*_{di})$ , in an arbitrary direction, is also bidimensional and the coefficient is the same as in Eq. (55), where now  $\varphi$  is the angle between the relative velocity and the  $x^*_3$  axis. It is also clear that, in comparison with the sphere's lift force, neither forces in other directions nor changes in sign arise. Since the  $1/2$  value for the sphere's case is reached asymptotically for both coefficients (see Figure- 2), the use of the sphere value is acceptable in any situation where bubbles are not subject to strong deformation forces. However, if departure from the spherical shape is important, for example when bubbles are subject to a strong shear, or acceleration is not negligible, the shape effect becomes important. For instance, a bubble with  $b=2c$  has  $\xi_0=1/3^{1/2}$ , which implies  $\beta/(2-\beta) \approx 0.31$  and  $(1-\beta)/\beta \approx 1.15$ , and the predictions are off approximately by a factor 2 , from that of a sphere.

The coefficients  $((1-\beta)/\beta - 1/2)$  and  $(1-2\beta/(2-\beta))$  can also be seen in

Figure- 2. As for the  $\omega^*_{0i}$  terms, the coefficient corresponding to the relative velocity in

the direction  $i=3$  is the bigger. As expected, the component  $\omega_{d3}^*$  does not contribute to the lift force. It must be noticed that depending upon the relative values of  $\omega_{di}^*$ ,  $\omega_{oi}^*$  and  $\beta$  the dominating terms might be the ones originated by the ellipsoid rotation,  $\omega_{di}^*$ . These terms could be important in geometries of practical interest, such as a sudden expansion where viscous effects will make the deformed bubble rotate.

## 5. Conclusions.

The virtual mass and the lift force for a rotating oblate ellipsoid moving in a inviscid fluid undergoing an unsteady translation, constant strain and constant rotation, were calculated. The lift force includes the contribution of the ellipsoid's rotation. The coefficients of virtual mass and lift force differ appreciably from those of the sphere only for ellipsoids of considerable eccentricity. Thus, it is acceptable to use the sphere coefficients when modeling two-phase flow regimes with no important deformation effects. In contrast, whenever there are significant deformation effects in bubbles shape, it is more appropriate to use the coefficients obtained in this work.

One of the more striking results is that the virtual mass and lift force coefficients for a given relative velocity direction are the same, although there is no evident reason for this to occur. It seems worth it to investigate this issue further.

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## 6. Appendix: Properties of oblate coordinates and Legendre functions definitions.

Unless explicitly stated, all definitions were taken from the work of Morse and Feshbach (1953).

Scale factors:

$$h_\xi = a \sqrt{\frac{\xi^2 + \eta^2}{\xi^2 + 1}}$$

$$h_\eta = a \sqrt{\frac{\xi^2 + \eta^2}{1 - \eta^2}}$$

$$h_\theta = a \sqrt{\xi^2 + 1} \sqrt{1 - \eta^2}$$

Gradient:

$$\nabla = \frac{1}{h_\xi} \frac{\partial}{\partial \xi} \tilde{\xi} + \frac{1}{h_\eta} \frac{\partial}{\partial \eta} \tilde{\eta} + \frac{1}{h_\theta} \frac{\partial}{\partial \theta} \tilde{\theta}$$

where  $\vec{\xi}$ ,  $\vec{\eta}$  and  $\vec{\theta}$  are the unit vectors normal to the surfaces  $\xi = \xi_0$ ,  $\eta = \eta_0$  and  $\theta = \theta_0$ , respectively, where  $\xi_0$ ,  $\eta_0$  and  $\theta_0$  are constants.

Area differential over the ellipsoid's surface:

$$dS = h_\theta h_\eta d\theta d\eta$$

Legendre Functions:

Functions of a real argument:

$$P_1^0(\eta) = \eta$$

$$P_1^1(\eta) = \sqrt{1 - \eta^2}$$

$$P_2^1(\eta) = 3\eta \sqrt{1 - \eta^2}$$

Functions of an imaginary argument, as redefined in this work to avoid imaginary numbers:

$$P_1^0(i\xi) = \xi$$

$$P_1^1(i\xi) = \sqrt{1 + \xi^2}$$

$$P_2^1(i\xi) = 3\xi \sqrt{1 + \xi^2}$$

$$Q_1^0(i\xi) = \xi \arctan(\frac{1}{\xi}) - 1$$

$$Q_1^1(i\xi) = \frac{\xi}{\sqrt{1 + \xi^2}} - \sqrt{1 + \xi^2} \arctan(\frac{1}{\xi})$$

$$Q_2^1(i\xi) = \frac{3\xi^2 + 2}{\sqrt{1 + \xi^2}} - 3\xi \sqrt{1 + \xi^2} \arctan(\frac{1}{\xi})$$

## **7. References.**

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**Figure-1 :Description of the problem to be solved. Vectors or tensors are bold.**

**Figure- 2: The coefficients of virtual mass and lift force (see Eqs. (53) and (57)) as a function of**

$$\xi_0 = \left[ \left( \frac{b}{c} \right)^2 - 1 \right]^{1/2} \quad \text{where } b \text{ and } c \text{ are the major and minor semiaxes respectively and}$$

$$\beta = \xi_0 \left( \xi_0^2 + 1 \right) \arctan \left( \frac{1}{\xi_0} \right) - \xi_0^2 \quad (\text{Eq. (50)}). \text{ Notice that the coefficients that tend to zero}$$

**correspond to the contribution of the ellipsoid rotation, } $\omega^*$ , to the lift force.**

$$v^*, e^*x^* + \omega_e^* x x^*$$

$$b > c$$

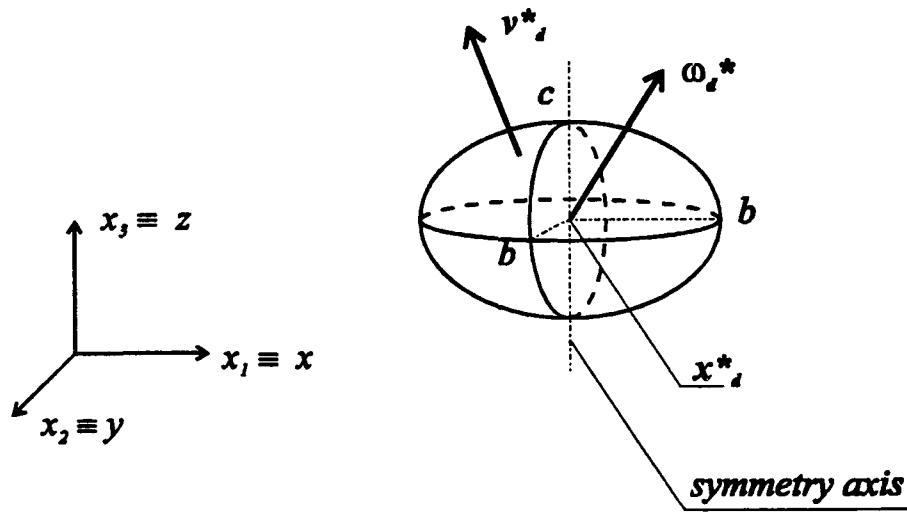


figure-1

figure-2

